Set point control in the state space setting

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Abstract

This report is intented as a supplement or an extension to the material used in connection to or after the courses **Stochastic Adaptive Control (02421)** and **Static and Dynamic Optimization (02711)** given at the **Department of Informatics and Mathematical Modelling, The Technical University of Denmark.**

The focus is in this report related to the problem of handling a set point or a constant reference in a state space setting. In principle just about any (state space control) design methodology may be applied. Here the presentation is based on LQ design, but other types such as poleplacement can be applied as well.

This is the Monte Petriolo paper which is a compilation of results gathered from the litterature. A major part of the results are collected from the basic control litterature during a sabatical year at Oxford university and is further compiled and reported in Monte Petriolo (Umbria, Italy).

1 Introduction

The focus in this report is set point control or control of a dynamic system with a piece wise constant reference. This problem is sparsely handled in the literature despite its practical application. This is obviously due to its lack of theoretical contents and interest.

We will in this report try to give the simplest presentation and illustrate the extention including more general frameworks. For that matter we will in the first part of the report assume that system is scalar (SISO or single input single output).

The reference is in this report assumed to be constant or piece wise constant, which off course include the standard step change in the reference. This problem has two major interest. The first problem is the regulation problem in which the set point is constant and the problem is to match the setpoint. The next focus area is the closed loop properties in connection to setpoint changes. This is the servo problem in connection to a step changes.

Related problem which are not addressed in this report are problems related to constant (or piece wise constant) disturbances. Moreover, constraints in the control actions (or related signals) or states are also omitted in this report. Obviously, other types of variations (eg. harmonic) in the reference signal are also omitted here.

In this report we will consider the problem of controlling a system given as:

$$x_{i+1} = Ax_i + Bu_i \qquad x_0 = \underline{x}_0$$

such that output

$$y_i = Cx_i + Du_i$$

match a set point (w). Often is it not only the output that enters into the objective function. In general we can focus on

$$z_i = C_z x_i + D_z u_i$$

The choice $C_z = I$, $D_z = 0$ is quite frequent. In the following n_x will denote the system order and the number of state, n_u is the number of inputs and n_y is the number of outputs.

2 The standard regulation problem

The object in regulation is to reduce the influence from disturbances or simple to keep the system close to the origin.

The standard regulation problem has several formulations. The most common version is an LQ formulation in which the cost function is quadratic in the states and the control actions. In the H_2 formulation the cost function is also a quadratic cost, but in an augmented output vector which contains elements of the errors (objective) and the control actions (costs).

2.1 The LQ formulation

The standard LQ problem is to find an input sequence u_i that take the system

$$x_{i+1} = Ax_i + Bu_i \qquad x_0 = \underline{x}_0$$

from an initial state x_0 along a trajectory such that the cost index

$$J = \sum_{i=0}^{N} x_i^T Q x_i + u_i^T R u_i$$

is minimized. Eventually the horizon, N, is infinite i.e. $N = \infty$. The solution to this problem (see [2]) can be formulated as a state feedback solution

$$\boxed{u_i = -Lx_i} \qquad (1)$$

or as a time function

$$u_i = -L\Phi^i x_0 \qquad \Phi = A - BL$$

depending only on the initial state and age (i.e. i). In a deterministic setting the two solutions are identical, but in an uncertain environment the first (the state feed back version) is more robust with respect to uncertainty and noise. The results can be found in Appendix D.1.

The LQ results are not restricted to the standard presentation given above. It can, as indicated in Appendix D.2, also include cross terms in the cost function.

$$J = \sum_{i=0}^{N} x_i^T Q x_i + u_i^T R u_i + 2x_i^T \mathbb{S} u_i = \sum_{i=0}^{N} \left[\begin{array}{cc} x_i^T & u_i^T \end{array} \right] \left[\begin{array}{cc} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{array} \right] \left[\begin{array}{c} x_i \\ u_i \end{array} \right]$$

The results are quite similar to the ones in (1) except off course for the dependence on $\mathbb{S}$.

2.2 Output control

Often the problem is to control the output of a system

$$y_i = C x_i$$

with due respect to the control effort. This is often formulated as a (LQ) problem in which the cost functions

$$J = \sum_{i=0}^{N} y_i^2 + \rho^2 u_i^2 \tag{2}$$

has to be minimized. Here ρ^2 is a design parameter. In the LQ framework this is equivalent to

$$Q = C^T C \qquad R = \rho^2$$

This problem also emerge if we want to minimize

$$J = \sum_{i=0}^{N} y_i^2 \quad \text{subject to} \quad \sum_{i=0}^{N} u_i^2 \leq C_u$$

where C_u is the design parameter. In that case ρ^2 is a Lagrange multiplier. In practice the two approaches are much more related than expected by a first inspection.

The problem in (2) can also be formulated as a minimization of

$$J = \sum_{i=0}^{N} \|z_i\|_q^2$$

where

$$z_i = \left[\begin{array}{c} y_i \\ u_i \end{array} \right] \qquad q = \left[\begin{array}{cc} 1 & 0 \\ 0 & \rho^2 \end{array} \right]$$

In general formulation of the H_2 problem is a minimization of the cost function

$$J = \sum_{i=0}^{N} \|z_i\|_q^2$$

where

$$z_i = C_z x_i + D_z u_i$$

There is a tight connection between $C_z,\ D_z,\ q$ and $Q,\ R,\ \mathbb{S}$. For example, the cost function in (2) appears when

$$C_z = \begin{bmatrix} C \\ 0 \end{bmatrix} \qquad D_z = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad q = \begin{bmatrix} 1 & 0 \\ 0 & \rho^2 \end{bmatrix}$$

In general the H_2 problem can be transformed into the LQ problem through:

$$\boxed{\begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} = \begin{bmatrix} C_z^T \\ D_z^T \end{bmatrix} q \begin{bmatrix} C_z & D_z \end{bmatrix}}$$

On the other hand the H_2 problem can emerge from the LQ problem if

$$\mathcal{Q} = \begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix}$$

is factorized into

$$\boxed{\mathcal{Q} = \begin{bmatrix} C_z^T \\ D_z^T \end{bmatrix} q \begin{bmatrix} C_z & D_z \end{bmatrix}}$$

This factorization is by no means unique. A factor can for example either be in q or in C_z and D_z.

In continuous time the problem is to control the system

$$\dot{x}_t = Ax_t + Bu_t \qquad x_0 = \underline{x}_0$$

such that the cost function

$$J = \frac{1}{2} x_T^T P x_T + \frac{1}{2} \int_0^T x_t^T Q x_t + u_t^T R u_t \ dt$$

is minimized. The solution to this problem is in stationarity ($T \to \infty$) given by:

$$\boxed{u_t = -Lx_t}$$

where:

$$0 = SA + A^T S + Q - SBR^{-1}B^T S$$

and

$$L = R^{-1}B^T S$$

If the cost function has a cross term

$$J = x_T^T P x_T + \int_0^T x_t^T Q x_t + x_t^T Q x_t + 2x_t^T \mathbb{S} u_t \ dt$$

then the solution is

$$\boxed{u_t = -R^{-1}(B^T S + \mathbb{S}^T)x_t}$$

where:

$$0 = S_t A + A^T S_t + Q - (S_t B + \mathbb{S})R^{-1}(B^T S_t + \mathbb{S}^T)$$

The realtion between the LQ and H_2 problem is the same in continuous time as in discrete time.

$$\begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} = \begin{bmatrix} C_z^T \\ D_z^T \end{bmatrix} W \begin{bmatrix} C_z & D_z \end{bmatrix}$$

As in the discrete time case this will typically be the case if the problem arise from a minimization of

$$J = \int_0^T |y_t|_W^2 \ dt$$

which is a weighted (W) integral square of the output

$$y_t = C_z x_t + D_z u_t$$

i.e. the H_2 problem. □

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3 Feed forward

Let us now turn the the focus area of this report, namely control of a system such that its output vector match the reference vector. We assume that a set point w_i exists, is piecewise constant and the dimension equals n_x.

Probably the most obvious way of introducing the reference (see eg. [1]) is to include a feed forward term ($M,\ n_u \times n_y$) in the control. That is to use control law

$$u_i = Mw_i - Lx_i$$

In this case the closed loop becomes

$$x_{i+1} = \Phi x + BMw_i$$

where

$$\Phi = A - BL$$

The feed forward term, M, can be chosen such that the output match the reference in stationarity, i.e.

$$y = \big[C(I-\Phi)^{-1}B + D\big]Mw$$

Let us denote the DC gain through the system as

$$\kappa = C(I-\Phi)^{-1}B + D$$

and assume that it is non zero (i.e. invertible). In the normal case where $n_u = n_y$ we can use:

$$M = \kappa^{-1}$$

If $n_u > n_y$ we have an extra flexibility and can use

$$M = \kappa^T(\kappa\kappa^T)^{-1}$$

On the other hand if $n_u < n_u$ we can't fulfill our objective. If we will minimize the distance between our objective (w) and our possibility (KMw) then we can use:

$$M = (\kappa^T\kappa)^{-1}\kappa^T$$

It is well known that this solution is sensitive to modelling errors. If the DC gain in the system model is wrong, then the closed loop will have a similar error.

In continuous time the discription is

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

and we have the same result just is the DC gain is given by

$$\kappa = CA^{-1}B + D$$

$\square$

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4 Target values

Another very classical way of handling the set point is to consider the standard problem as a deviation away from a stationary point. This method has been extensively been used in [3]. We will the denote this point as a target point. That is to consider the problem as a redefinition of origin according to the reference. That means the control is given by:

$$u_i = u_0 - L(x_i - x_0)$$

where the target point $x_0, \;\; u_0$ has to be a stationary point i.e. to fulfill

$$x_0 = Ax_0 + Bu_0$$

The target point also have to match the set point, i.e. to fulfill

$$w = Cx_0 + Du_0$$

In total the stationary point should satisfy the following equation

$$\begin{bmatrix} A-I & B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} w$$

This indicates that both x_0 and u_0 is proportional to the set point and

$$Mw = u_0 + Lx_0$$

where M was discussed in the previous section. This solution has, as the feed forward method, a high sensitivity to modelling errors. The classical way of handling this problem is use integral action as we will return to in Section 9. Here we will pursuit another type of approach.

If we look at the problem of modelling errors, we can handle it in stationarity. One way to tackle the problem is to include a (constant) disturbance in the input or the output.

$$\begin{aligned} x_{i+1} &= Ax_i + B(u_i + d) \\ y_i &= Cx_i + D(u_i + d) \end{aligned}$$

The input disturbance has be estimated with an observer or a Kalman filter. In that way, the disturbance will express the modelling error at DC.

Another, but similar, approach is to include en output disturbance and operate with a design model

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i \\ y_i &= Cx_i + Du_i + d \end{aligned}$$

Also here the disturbance has to be estimated.

The two approaces can be merged into the general form

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i + Gd \\ y_i &= Cx_i + Du_i + Hd \end{aligned} \tag{3}$$

where G and H are matrices of appropriate dimensions. In that case we can find the target values from:

$$\begin{bmatrix} A-I & B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} + \begin{bmatrix} G \\ H \end{bmatrix} d = \begin{bmatrix} 0 \\ I \end{bmatrix} w$$

where d has to be estimated.

If we assume that the disturbance closely is constant we can model it by the following model

$$d_{i+1} = d_i + \xi_i \qquad \xi_i \in \mathbf{N}_{iid}(0, R_d)$$

and augment the stochasic version of (3).

$$\begin{aligned} \begin{bmatrix} x \\ d \end{bmatrix}_{i+1} &= \begin{bmatrix} A & G \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix}_i + \begin{bmatrix} B \\ 0 \end{bmatrix} u_i + \begin{bmatrix} v \\ \xi \end{bmatrix} \\ y_i &= \begin{bmatrix} C & H \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix}_i + Du_i + e_i \end{aligned} \tag{4}$$

then d (and state x) can be estimated by means of an observer or a Kalman filter.

In continuous time the system is described by

$$\frac{d}{dt}x = Ax + Bu + Gd$$
$$y = Cx + Du + Hd$$

and the target values can be determined by:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} + \begin{bmatrix} G \\ H \end{bmatrix} d = \begin{bmatrix} 0 \\ I \end{bmatrix} w$$

The state variable which is not neccesarely known can be estimated from

$$\frac{d}{dt}d = \xi$$

the description

$$\frac{d}{dt}\begin{bmatrix} x \\ d \end{bmatrix} = \begin{bmatrix} A & G \\ 0 & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t + \begin{bmatrix} v \\ \xi \end{bmatrix}_t$$

□

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5 Optimal tracking

In this section we will extend the standard discrete time LQ control problem to include a reference signal. This approach is based on the presentation and results in [2]. In this presentation we wil restrict the approach and assume that the reference signal is constant (ie. is a set point). Consider the problem of controlling a dynamic system in discrete time

$$x_{i+1} = Ax_i + Bu_i \qquad x_0 = \underline{x}_0 \tag{5}$$

such that the cost function

$$J = \|y_N - r_N\|_P^2 + \sum_{i=0}^{N-1} \|Cx_i - r_i\|_Q^2 + \|u_i\|_R^2$$

is minimized.

In steady state ($N \to \infty$, $r_i = r$) the solution is given by

$$u_i = -Lx_i + Kv$$

where

$$L = \left[R + B^T SB\right]^{-1} B^T SA \qquad\qquad K = \left[R + B^T SB\right]^{-1} B^T$$

and S is the solution to the algebraic Ricatti equation;

$$S = C^T Q C + A^T S A - A^T S B \left[R + B^T S B\right]^{-1} B^T S A$$

and v found from

$$v = (I - \Phi^T)^{-1} C^T Q r \quad \text{where} \quad \Phi = A - BL$$

Unless the system has (or has been imposed) a pure integration, this strategy will result in a steady state error (for $R > 0$).

In continuous time the problem is to control the system

$$\dot{x}_t = A x_t + B u_t \qquad x_0 = \underline{x}_0$$

such that the cost function

$$J = \|y_T - r_T\|_P^2 + \int_0^T \|C x_t - r_t\|_Q^2 + \|u_t\|_R^2 \; dt$$

is minimized. In steady state ($T \to \infty$, $r_t = r$) the solution is given by

$$u_t = -L x_t + K v$$

where

$$L = R^{-1} B^T S_t \qquad\qquad K = R^{-1} B^T$$

and S is the solution to the algebraic Ricatti equation;

$$0 = SA + A^T S + C^T Q C - S B R^{-1} B^T S$$

and v found from

$$v = \Phi^{-T} C^T Q r \quad \text{where} \quad \Phi = A - BL$$

Consider the problem of controlling a dynamic system in continuous time given by (20) such that the cost function

$$J = \|\bar{C} x_T - r_T\|_P^2 + \int_0^T \|z_t - r_t\|_Q^2 \; dt$$

where

$$z_t = C x_t + D u_t$$

is minimized.

In steady state ($T \to \infty$) the solution is given by

$$u_t = -L x_t + K v + M r$$

where

$$L_t = \left[D^T Q D\right]^{-1} (D^T Q C + B^T S_t)$$

$$K_t = \left[D^T Q D\right]^{-1} B^T \qquad\qquad M_t = \left[D^T Q D\right]^{-1} D^T Q$$

and S is the solution to the algebraic Ricatti equation;

$$0 = S_t A + A^T S_t + C^T Q C - (C^T Q D + S B)(D^T Q D)^{-1} (D^T Q C + B^T S_t)$$

and v found from

$$v = \Phi^{-T} (C - DL)^T Q r$$

□

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6 Internal model principle (IMP)

Yet another way of transforming the set point problem (and other type reference problems) into a standard regulation problem is to use the socalled Internal Model Principle (IMP). I that case a model of the set point variance is created and build into an augmented system description.

The model

$$r_{i+1} = r_i + \xi_i$$

where ξ_i is a white noise sequence, is a suitable model for set points variation (with unknown changes and unknown instants of changes). In that case the total description becomes

$$\begin{bmatrix} x \\ r \end{bmatrix}_{i+1} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix}_i + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xi_i$$

$$y_i = \begin{bmatrix} -C & 1 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix}_i$$

One obvious problem in this approach is that the system description is not controlable. That often results often results in problems when using standard software for solving the Ricatti equation. The solution exists due to the fact that the uncontrollable part of the state space is not visible in the cost function. The solution

$$u_i = -\begin{bmatrix} L_x & L_z \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix}_i$$

can be compared with the solution in section 5.

In continuous time the system

$$\frac{d}{dt} x_t = A x_t + B u_t \qquad\qquad x_0 = \underline{x}_0$$

has to be controlled such the output is close to the reference. The model

$$\frac{d}{dt} r_t = \xi_t$$

where ξ_t is white noise is a suitable model for a setpoint signal. The total model can be given as:

$$\frac{d}{dt} \begin{bmatrix} x \\ r \end{bmatrix}_t = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix}_t + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xi_t$$

and the task is to minimize the error

$$y_t = \begin{bmatrix} -C & 1 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix}_t$$

(in some sense and with respect to the control power). The solution is

$$u_t = -\begin{bmatrix} L_x & L_z \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix}_t$$

which is a feedback from the state and a feedforward from the set point signal. □

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7 Control moves

In this section we will discuss control formulated in terms control moves. This can be regarded as if the decision variable is the velocity of the control. It can be extended to include the derivative of the control action to any order.

If an optimal control strategy from section 5 or 6 is applied on a system (without an integral action) then a non zero set point will result in a steady state error. This is due a conflict between steady state error and steady state control action. One remedy is to discharge the DC component of the control action in the cost function. The most direct and simples way of doin this is to consider the control moves (control velocity) rather than the control action itself.

7.1 Velocity control

The problem of controlling a dynamic system

$$x_{i+1} = Ax_i + Bu_i$$

$$y_i = Cx_i$$

such that the output is close to a certain (constant) set point, r, has some challenges. In order to avoid the problem of a steady state error we can formulate the objective in terms of the control moves v_i. If the control moves are constant between samples then:

$$u_i = z_i + T_s v_i \qquad \text{where} \qquad z_{i+1} = z_i + T_s v_i$$

and the description of the system can be written as:

$$\begin{bmatrix} x \\ z \end{bmatrix}_{i+1} = \begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}_i + \begin{bmatrix} BT_s \\ T_s \end{bmatrix} v_i \tag{6}$$

$$y_i = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}_i$$

This is more or less the same as introducing (a discrete time) integration in front of the system. The problem is then to control the system in (6) according to the method described in section 5 and 6. In that case the cost is changed and the control related cost in the objective function is shifted in frequency and the weight is put on the control velocity. The control action can also be

7.2 Acceleration control

The method can be extended and the problem formulated such that the decision variable is the acceleration, a_i, then (if the acceleration is constant between samples):

$$u_i = z_i + T_s v_i + \frac{1}{2}T_s^2 a_i$$

where

$$z_{i+1} = z_i + T_s v_i + \frac{1}{2}T_s^2 a_i \qquad v_{i+1} = v_i + T_s a_i$$

and the system can be written as:

$$\begin{bmatrix} x \\ z \\ v \end{bmatrix}_{i+1} = \begin{bmatrix} A & B & BT_s \\ 0 & 1 & T_s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \\ v \end{bmatrix}_i + \begin{bmatrix} \frac{1}{2}BT_s^2 \\ \frac{1}{2}T_s^2 \\ T_s \end{bmatrix} a_i \tag{7}$$

$$y_i = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ v \end{bmatrix}_i$$

This is the same as introducing two integration in front of the system.

7.3 General velocity control

These ideas can be extended to just about any order. Let us illustrate this in a forth order case. Let u be the control action and let u_4 be the forth order derivative. Then (if u_4 is constant between samples):

$$u = z + T_s u_1 + \frac{1}{2}T_s^2 u_2 + \frac{1}{6}T_s^3 u_3 + \frac{1}{24}T_s^4 u_4$$

where

$$\begin{bmatrix} z \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}_{i+1} = \begin{bmatrix} z + T_s u_1 + \frac{1}{2}T_s^2 u_2 + \frac{1}{6}T_s^3 u_3 + \frac{1}{24}T_s^4 u_4 \\ u_1 + T_s u_2 + \frac{1}{2}T_s^2 u_3 + \frac{1}{6}T_s^3 u_4 \\ u_2 + T_s u_3 + \frac{1}{2}T_s^2 u_4 \\ u_3 + T_s u_4 \end{bmatrix}$$

or stated otherwise:

$$\begin{bmatrix} z \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}_{i+1} = \begin{bmatrix} 1 & T_s & \frac{1}{2}T_s^2 & \frac{1}{6}T_s^3 \\ 0 & 1 & T_s & \frac{1}{2}T_s^2 \\ 0 & 0 & 1 & T_s \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}_i + \begin{bmatrix} \frac{1}{24}T_s^4 \\ \frac{1}{6}T_s^3 \\ \frac{1}{2}T_s^2 \\ T_s \end{bmatrix} u_4$$

The total system description becomes:

$$\begin{bmatrix} x \\ z \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}_{i+1} = \begin{bmatrix} A & B & BT_s & \frac{1}{2}BT_s^2 & \frac{1}{6}BT_s^3 \\ 0 & 1 & T_s & \frac{1}{2}T_s^2 & \frac{1}{6}T_s^3 \\ 0 & 0 & 1 & T_s & \frac{1}{2}T_s^2 \\ 0 & 0 & 0 & 1 & T_s \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}_i + \begin{bmatrix} \frac{1}{24}BT_s^4 \\ \frac{1}{24}T_s^4 \\ \frac{1}{6}T_s^3 \\ \frac{1}{2}T_s^2 \\ T_s \end{bmatrix} u_4$$

$$y_i = \begin{bmatrix} C & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}_i$$

This method can be extented to any order.

7.4 Generelized velocity control

In the previuous sections we solved the problem of matching the set point by means of using a higher order derivative as the decision variable. In this section we will use a combinations of higher order derivatives as decision variables. That means the control decision is a blend of different frequencies.

7.4.1 part 1

Let us first focus on the problem when we use the control move, v_i and the control level, $\tilde{u}_i$, as decision variable. In that case:

$$u_i = \tilde{u}_i + z_i + T_s v_i \quad \text{where} \quad z_{i+1} = z_i + T_s v_i$$

or simply:

$$\begin{bmatrix} x \\ z \end{bmatrix}_{i+1} = \begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}_i + \begin{bmatrix} B & BT_s \\ 0 & T_s \end{bmatrix} \begin{bmatrix} \tilde{u} \\ v \end{bmatrix}_i$$

$$y_i = \begin{bmatrix} C \; 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}_i$$

7.4.2 Part 2

If the decision variables are the acceleration, the level of the control move (velocity) and the control level, then we can introduce

$$v_{i+1} = v_i + \tilde{v}_i + T_s a_i$$

$$z_{i+1} = z_i + T_s(v_i + \tilde{v}_i) + \frac{1}{2}T_s^2 a_i$$

where $\tilde{u}_i$ and $\tilde{v}_i$ are perturbations on the control and control move respectively. The decision consists of the vector

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ a \end{bmatrix}_i$$

In that case:

$$u_i = \tilde{u}_i + z_i + T_s(v_i + \tilde{v}_i) + \frac{1}{2}T_s^2 a_i$$

Consequently, the acceleration model in (7) can be transformed into

$$\begin{bmatrix} x \\ z \\ v \end{bmatrix}_{i+1} = \begin{bmatrix} A & B & BT_s \\ 0 & 1 & T_s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \\ v \end{bmatrix}_i + \begin{bmatrix} B & BT_s & \frac{1}{2}BT_s^2 \\ 0 & T_s & \frac{1}{2}T_s^2 \\ 0 & 1 & T_s \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ a \end{bmatrix}_i \tag{8}$$

$$y_i = \begin{bmatrix} C \ 0 \ 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ v \end{bmatrix}_i$$

7.4.3 Part 3

The method can, as the simple method from the previous sections, be extended to any finite order. For $n = 4$ we have the resulting system description:

$$\begin{bmatrix} x \\ z \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}_{i+1} = \begin{bmatrix} A & B & BT_s & \frac{1}{2}BT_s^2 & \frac{1}{6}BT_s^3 \\ 0 & 1 & T_s & \frac{1}{2}T_s^2 & \frac{1}{6}T_s^3 \\ 0 & 0 & 1 & T_s & \frac{1}{2}T_s^2 \\ 0 & 0 & 0 & 1 & T_s \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}_i$$

$$+ \begin{bmatrix} B & BT_s & \frac{1}{2}BT_s^2 & \frac{1}{6}BT_s^3 & \frac{1}{24}BT_s^4 \\ 0 & T_s & \frac{1}{2}T_s^2 & \frac{1}{6}T_s^3 & \frac{1}{24}T_s^4 \\ 0 & 1 & T_s & \frac{1}{2}T_s^2 & \frac{1}{6}T_s^3 \\ 0 & 0 & 1 & T_s & \frac{1}{2}T_s^2 \\ 0 & 0 & 0 & 1 & T_s \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ u_4 \end{bmatrix}$$

$$y_i = \begin{bmatrix} C & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}_i$$

In continuous time the problem is to control the system

$$\begin{aligned} \frac{d}{dt}x_t &= Ax_t + Bu_t & \qquad x_0 = \underline{x}_0 \\ y &= Cx_t \end{aligned}$$

such that the output match the reference.

Velocity control: If we want to use the control velocity

$$\frac{d}{dt}u_t = v_t$$

rather than the control signal itself as the decision variable then the system can be described as:

$$\frac{d}{dt}\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_t$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}_t$$

Acceleration control: If we want to use the control acceleration

$$\frac{d}{dt}u_t = v_t \qquad \frac{d}{dt}v_t = a_t$$

as the decision variable then the system can be described as:

$$\frac{d}{dt}\begin{bmatrix} x \\ u \\ v \end{bmatrix} = \begin{bmatrix} A & B & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ u \\ v \end{bmatrix}_t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} a_t$$

$$y = \begin{bmatrix} C & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ u \\ v \end{bmatrix}_t$$

General velocity control: As in descrete time these idears can be generalized to any order. Let the decision variable be

$$v = u^{(n)}$$

where

$$u^{(n)} = \frac{d^n}{dt^n}u$$

Then for $n = 4$ we have the description:

$$\frac{d}{dt}\begin{bmatrix} x \\ u \\ u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{bmatrix} = \begin{bmatrix} A & B & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ u \\ u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{bmatrix}_t + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v_t$$

$$y = \begin{bmatrix} C & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ u \\ u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{bmatrix}_t$$

Generalized velocity control:

If we will use a high order derivative as well as pertubations of its integrals, we can use the following agmentation scheme. The for $n = 4$ we have the description:

$$\frac{d}{dt}\begin{bmatrix} x \\ u \\ u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{bmatrix} = \begin{bmatrix} A & B & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ u \\ u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{bmatrix} + \begin{bmatrix} B & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{u} \\ \tilde{u}^{(1)} \\ \tilde{u}^{(2)} \\ \tilde{u}^{(3)} \\ \tilde{u}^{(4)} \end{bmatrix}$$

$$y = \begin{bmatrix} C & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ u \\ u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{bmatrix}_t$$

□

8 Frequency weighting

The reason for introducing velocity control and its related is to let the control signal enters into the performance in such a way that the resulting controller can compensate for the setpoint change in an appropriate manner.

Now assume the the signals in the performance index are filtered version of the original signals, such as the output, y_t, control u_t.

$$\bar{J} = E\left\{\sum_{t=0}^{N} \begin{pmatrix} y_t^{f\top} & u_t^{f\top} \end{pmatrix} \begin{bmatrix} O_1 & O_{12} \\ O_{12}^\top & O_2 \end{bmatrix} \begin{pmatrix} y_t^f \\ u_t^f \end{pmatrix}\right\}$$

where

$$y_t^f = H_y(q) y_t \qquad u_t^f = H_u(q) u_t$$

These frequency weights or filters has a state space representation such as:

$$x_{t+1}^y = A^y x_t^y + B^y y_t \qquad x_{t+1}^u = A^u x_t^u + B^u u_t$$

$$y_t^f = C^y x_t^y + D^y y_t \qquad u_t^f = C^u x_t^u + D^u u_t$$

If the system is given by:

$$x_{i+1} = A x_i + B u_i$$

$$y = C x_i + D u_i$$

then the system description can be augmented to include the filtered versions of the signals:

$$\boxed{\begin{bmatrix} x \\ x^y \\ x^u \end{bmatrix}_{t+1} = \begin{bmatrix} A & 0 & 0 \\ B^y C & A^y & 0 \\ 0 & 0 & A^u \end{bmatrix} \begin{bmatrix} x \\ x^y \\ x^u \end{bmatrix}_t + \begin{bmatrix} B \\ B^y D \\ B^u \end{bmatrix} u_t}$$

For the filthered output we have

$$\begin{aligned} y_t^f &= D^y C x_t + C^y x_t^y + D^y D u_t \\ &= \begin{pmatrix} D^y C & C^y & 0 & D^y D \end{pmatrix} \begin{bmatrix} x_t \\ x_t^y \\ x_t^u \\ u_t \end{bmatrix} \end{aligned}$$

In a similar way is

$$u_t^f = \begin{pmatrix} 0 & 0 & C^u & D^u \end{pmatrix} \begin{bmatrix} x_t \\ x_t^y \\ x_t^u \\ u_t \end{bmatrix}$$

If the augmented state vector

$$\bar{x}_t = \begin{bmatrix} x_t \\ x_t^y \\ x_t^u \end{bmatrix}$$

is introduced the we can write:

$$\boxed{\begin{bmatrix} y_t^f \\ u_t^f \end{bmatrix} = \begin{pmatrix} D^y C & C^y & 0 & D^y D \\ 0 & 0 & C^u & D^u \end{pmatrix} \begin{bmatrix} \bar{x}_t \\ u_t \end{bmatrix}}$$

Instead of minimizing the original cost we can minimize

$$\bar{J} = E\left\{\sum_{t=0}^{N} \left(\bar{x}_t^\top \;\; u_t^\top\right) \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^\top & Q_2 \end{bmatrix} \begin{pmatrix} \bar{x}_t \\ u_t \end{pmatrix}\right\}$$

where

$$\boxed{\begin{gathered}\begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^\top & Q_2 \end{bmatrix} = \\ \begin{bmatrix} C^T(D^y)^T & 0 \\ (C^y)^T & 0 \\ 0 & (C^u)^T \\ D^T(D^y)^T & (D^y)^T \end{bmatrix} \begin{bmatrix} O_1 & O_{12} \\ O_{12}^\top & O_2 \end{bmatrix} \begin{bmatrix} D^yC & C^y & 0 & D^yD \\ 0 & 0 & C^u & D^u \end{bmatrix}\end{gathered}}$$

Now the continuous time version of the problem just mentioned. Assume the the signals in the performance index are filtered version of the original signals, such as the output, y_t, control u_t.

$$\bar{J} = E\left\{\int_{t=0}^{T} \left(y_t^{f\top} \;\; u_t^{f\top}\right) \begin{bmatrix} O_1 & O_{12} \\ O_{12}^\top & O_2 \end{bmatrix} \begin{pmatrix} y_t^f \\ u_t^f \end{pmatrix} dt\right\}$$

where

$$y_t^f = H_y(p)y_t \qquad u_t^f = H_u(p)u_t$$

These frequency weights or filters has a state space representation such as:

$$\begin{aligned} \frac{d}{dt}x^y &= A^y x_t^y + B^y y_t & \qquad \frac{d}{dt}x^u &= A^u x_t^u + B^u u_t \\ y_t^f &= C^y x_t^y + D^y y_t & \qquad u_t^f &= C^u x_t^u + D^u u_t \end{aligned}$$

If the system is given by:

$$\begin{aligned} \frac{d}{dt}x_t &= Ax_t + Bu_t \\ y_t &= Cx_t + Du_t \end{aligned}$$

then the system description can be augmented to include the filtered versions of the signals:

$$\boxed{\frac{d}{dt}\begin{bmatrix} x \\ x^y \\ x^u \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ B^yC & A^y & 0 \\ 0 & 0 & A^u \end{bmatrix} \begin{bmatrix} x \\ x^y \\ x^u \end{bmatrix}_t + \begin{bmatrix} B \\ B^yD \\ B^u \end{bmatrix} u_t}$$

For the filthered output we have

$$\begin{aligned} y_t^f &= D^yCx_t + C^yx_t^y + D^yDu_t \\ &= \begin{pmatrix} D^yC & C^y & 0 & D^yD \end{pmatrix} \begin{bmatrix} x_t \\ x_t^y \\ x_t^u \\ u_t \end{bmatrix} \end{aligned}$$

In a similar way is

$$u_t^f = \begin{pmatrix} 0 & 0 & C^u & D^u \end{pmatrix} \begin{bmatrix} x_t \\ x_t^y \\ x_t^u \\ u_t \end{bmatrix}$$

If the augmented state vector

$$\bar{x}_t = \begin{bmatrix} x_t \\ x_t^y \\ x_t^u \end{bmatrix}$$

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$$\boxed{\begin{bmatrix} y_t^f \\ u_t^f \end{bmatrix} = \begin{pmatrix} D^yC & C^y & 0 & D^yD \\ 0 & 0 & C^u & D^u \end{pmatrix} \begin{bmatrix} \bar{x}_t \\ u_t \end{bmatrix}}$$

Instead of minimizing the original cost we can minimize

$$\bar{J} = E\left\{\int_{t=0}^{T} \left(\bar{x}_t^\top \;\; u_t^\top\right) \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^\top & Q_2 \end{bmatrix} \begin{pmatrix} \bar{x}_t \\ u_t \end{pmatrix} dt\right\}$$

where

$$\boxed{\begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^\top & Q_2 \end{bmatrix} = \begin{bmatrix} C^T(D^y)^T & 0 \\ (C^y)^T & 0 \\ 0 & (C^u)^T \\ D^T(D^y)^T & (D^y)^T \end{bmatrix} \begin{bmatrix} O_1 & O_{12} \\ O_{12}^\top & O_2 \end{bmatrix} \begin{bmatrix} D^y C & C^y & 0 & D^y D \\ 0 & 0 & C^u & D^u \end{bmatrix}}$$

□

9 Integral action

Within the control area there are a few methods to avoid a steady state errors. One of the standard tricks in control is to introduce an integral action. In the state space setting it involves a state which integrate the error, ie.

$$z_{i+1} = z_i + (r_i - y_i)$$

That results in an augmented system given by

$$\begin{bmatrix} x \\ z \end{bmatrix}_{i+1} = \begin{bmatrix} A & 0 \\ -C & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}_i + \begin{bmatrix} B \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y_i = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}_i$$

If the controller is given by

$$u_i = u_0 - L \begin{bmatrix} x_i - x_0 \\ z_i - z_0 \end{bmatrix}$$

then the closed loop is given by

$$\begin{bmatrix} x \\ z \end{bmatrix}_{i+1} = \begin{bmatrix} A - BL_x & -BL_z \\ -C & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}_i + \begin{bmatrix} B \\ 0 \end{bmatrix} u_0 + \begin{bmatrix} B \\ 0 \end{bmatrix} L \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

This can also be formulated as:

$$u_i = Mr_i - L \begin{bmatrix} x_i \\ z_i \end{bmatrix}$$

which results in

$$\begin{bmatrix} x \\ z \end{bmatrix}_{i+1} = \begin{bmatrix} A - BL_x & -BL_z \\ -C & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}_i + \begin{bmatrix} BM \\ 1 \end{bmatrix} r$$

$$e = \begin{bmatrix} -C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}_i + r$$

The problem is to determine M such that the response is optimal.

Within the control area there are a few methods to avoid a steady state errors. One of the standard tricks in control is to introduce an integral action. In the state space setting it involves a state which integrate the error, ie.

$$\frac{d}{dt} z = r_t - y_t$$

That results in an augmented system given by

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 1 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}_i + \begin{bmatrix} B \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y_t = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}_t$$

If the controller is given by

$$u_t = u_0 - L\begin{bmatrix} x_t - x_0 \\ z_t - z_0 \end{bmatrix}$$

then the closed loop is given by

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix}_t = \begin{bmatrix} A - BL_x & -BL_z \\ -C & 1 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}_t + \begin{bmatrix} B \\ 0 \end{bmatrix} u_0 + \begin{bmatrix} B \\ 0 \end{bmatrix} L \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

This can also be formulated as:

$$u_t = Mr_t - L\begin{bmatrix} x_t \\ z_t \end{bmatrix}$$

which results in

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix}_t = \begin{bmatrix} A - BL_x & -BL_z \\ -C & 1 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}_t + \begin{bmatrix} BM \\ 1 \end{bmatrix} r$$

$$e = \begin{bmatrix} -C & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}_t + r$$

The problem is to determine M such that the response is optimal. □

10 Conclusion

In this paper we have reviwed different control methods for handling set points in a state space setting. The methods covers area from feed forward and target values to frequency weights and integral action. The paper is held in discrete time, but the results is also given in continuous time.

11 Acknowledgement

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A Quadratic optimization I

Consider the problem of minimizing a quadratic cost function

$$\begin{aligned} J &= \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} \\ h_{12}^T & h_{22} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\ &= x^T h_{11} x + \underline{2x^T h_{12}} u + \underline{u^T h_{22}} u \end{aligned}$$

It is quite elementary to find the derivative of the cost function

$$\frac{d}{dt} u J = 2x^T h_{12} + 2u^T h_{22} \tag{9}$$

and the stationary point must fulfill

$$h_{12}^T x + h_{22} u = 0$$

The stationary point

$$\boxed{u = -h_{22}^{-1} h_{12}^T x}$$

is a minimum to the cost function if h_{22} is possitive definite. Furthermore, the minimum of the cost function is quadratic in x. If we use (9) and *complete the square* then:

$$\begin{aligned} J^* &= x^T h_{11} x - (u^*)^T h_{22} u^* \\ &= x^T \left(h_{11} - h_{12} h_{22}^{-1} h_{12}^T \right) x \\ &= x^T S x \end{aligned}$$

where

$$\boxed{S = h_{11} - h_{12} h_{22}^{-1} h_{12}^T}$$

B Quadratic forms II

Let us now consider the more complex problem of minimizing

$$J = \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} \\ h_{12}^T & h_{22} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} g_x^T & g_u^T \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \sigma$$

or simply:

$$J = x^T h_{11} x + \underline{2x^T h_{12}} u + \underline{u^T h_{22}} u + g_x^T x + \underline{g_u^T} u + \sigma$$

A standard result gives the minimum as

$$2x^T h_{12} + 2u^T h_{22} + g_u^T = 0 \tag{10}$$

or

$$u = -h_{22}^{-1} [h_{12}^T x + \frac{1}{2} g_u]$$

Applying (10) gives the optimal cost:

$$\begin{aligned} J^* &= x^T h_{11} x + g_x^T x + \sigma - (u^*)^T h_{22} u^* \\ &= x^T (h_{11} - h_{12} h_{22}^{-1} h_{12}^T) x + (g_x^T - g_u^T h_{22}^{-1} h_{12}^T) x + \sigma - \frac{1}{4} g_u^T h_{22}^{-1} g_u \\ &= x^T \tilde{h}_{11} x + \tilde{g}_x^T x + \tilde{\sigma} \end{aligned}$$

where

$$\begin{array}{rcl} \tilde{h}_{11} & = & h_{11} - h_{12}h_{22}^{-1}h_{12}^T \\ \tilde{g} & = & g_x - h_{12}h_{22}^{-1}g_u \\ \tilde{\sigma} & = & \sigma - \frac{1}{4}g_u^T h_{22}^{-1}g_u \end{array}$$

C Feed forward

This appendix is a straith forward extention of section 3 and is valid both in a descrete and continuous time framework.

In closed loop and in stationarity we have

$$\bar{y} = K\bar{u}$$

where $K \in \mathbb{R}^{n_y \times n_u}$ is the DC gain through the system. Our objective is to find $\bar{u}$ such that

$$\boxed{\bar{y} = w}$$

In the feedforward strategy we use

$$\bar{u} = Mw$$

where $M \in \mathbb{R}^{n_u \times n_y}$.

C.1 The balanced problem

If $n_y = n_u$ and if K is nonsingular then it is trivial that

$$\bar{u} = K^{-1}w$$

or

$$\boxed{M = K^{-1}}$$

C.2 The overflexible problem

Firstly, let $n_u > n_y$. That means we have a surplus of flexibility to achieve our objective. We can then choice to find that stationary control which has the lowest size (and still achieve our objective). This can be formulated as minimizing

$$J = \frac{1}{2}\bar{u}^T\bar{u}$$

subject to

$$w = K\bar{u}$$

The Lagrange function for this problem is

$$J_L = \frac{1}{2}\bar{u}^T\bar{u} + \lambda^T(K\bar{u} - w)$$

which is stationary wrt. $\bar{u}$ for

$$\bar{u}^T + \lambda^T K = 0 \quad \text{or for} \quad \bar{u} = -K^T \lambda$$

In order to achieve our objective we must have

$$\lambda = -(KK^T)^{-1} w$$

or

$$\bar{u} = K^T (KK^T)^{-1} w$$

This results in

$$\boxed{M = K^T (KK^T)^{-1}}$$

C.3 The restricted problem

Let us then focus on the situation where $n_u < n_y$ ie. when we have less control flexibility. In that case we can't achieve our objective ($\bar{y} = w$) but have to find a $\bar{u}$ (or a M) such that the distance between the objective and the possible is minimized. In other word we will find an $\bar{u}$ such that

$$J = \frac{1}{2} \varepsilon^T \varepsilon \quad \text{where} \quad \varepsilon = w - K\bar{u}$$

is minimized. This is obtained for

$$I = -\varepsilon^T K = -(w - K\bar{u})^T K = 0$$

or if

$$\bar{u} = (K^T K)^{-1} K^T w$$

This results in:

$$\boxed{M = (K^T K)^{-1} K^T}$$

D The Discrete Time LQ control problem

In this section we will review the results related to control of a linear time invariant dynamic system

$$x_{i+1} = Ax_i + Bu_i \qquad x_0 = \underline{x}_0 \tag{11}$$

such that a quadratic cost function is minimized.

D.1 The Standard DLQ Control Problem

Let us first focus on the standard problem. In this context we will control the system in (11) such that the (standard LQ) cost function

$$J = x_N^T P x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i \tag{12}$$

is minimized. The Bellman equation will in this case be

$$V_i(x_i) = \min_{u_i} \left[x_i^T Q x_i + u_i^T R u_i + V_{i+1}(x_{i+1}) \right] \tag{13}$$

with the end point constraints

$$V_N(x_N) = x_N^T P x_N$$

If we test the candidate function

$$V_i(x_i) = x_i^T S_i x_i$$

then the inner part of the minimization in (13) will be

$$I = \begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} Q + A^T S_{i+1} A & A^T S_{i+1} B \\ B^T S_{i+1} A & R + B^T S_{i+1} B \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix}$$

The minimum for the this function is according to Appendix A given by

$$u_i = -L_i x_i \qquad L_i = \left[R + B^T S_{i+1} B\right]^{-1} B^T S_{i+1} A$$

and the candidate function is in fact a solution to the Bellman equation in (13) if

$$S_i = Q + A^T S_{i+1} A - A^T S_{i+1} B \left[R + B^T S_{i+1} B\right]^{-1} B^T S_{i+1} A \qquad S_N = P$$

If the gain, L_i, is used in the recursion for S_i

$$S_i = Q + A^T S_{i+1}(A - BL_i) \qquad S_N = P$$

As a simple implication from the proof we have that

$$V_0(x_0) = J^\star = x_0^T S_0 x_0$$

which is usefull in connection to an interpretation of S.

D.2 DLQ and cross terms

In order to connect the (very) related LQ formulation and H_2 formulation we will augment the standard problem with cross terms in the cost function. Assume that a discrete time (LTI) system is given as in (11) and the cost function (instead of (12)) is:

$$J = x_N^T P x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + 2 x_i^T \mathbb{S} u_i$$

or

$$J = x_N^T P x_N + \sum_{i=0}^{N-1} \begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix}$$

The situation becomes a bit more complicated. The cross terms especially occurs if the control problem is formulated as a problem in which (the square of) an output signal

$$y_i = C x_i + D u_i$$

is minimized, i.e.

$$J = \sum_{i=0}^{N-1} |y_i|_W^2$$

In that case

$$\begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} W \begin{bmatrix} C & D \end{bmatrix}$$

The Bellman equation becomes in the special case

$$V_i(x_i) = \min_{u_i}\left[\begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} + V_{i+1}(x_{i+1}) \right]$$

$$V_N(x_N) = x_N^T P x_N$$

and again we will try the follwoing candidate function

$$V_i(x_i) = x_i^T S_i x_i$$

This can be solved head on or by transforming the problem into the standard one.

D.2.1 Using transformation technique

If R is invertible then we can intoduce a new decision variable, v_i, given by:

$$u_i = v_i - R^{-1}\mathbb{S}^T x_i$$

The instanuous loss term (first term in the Bellman equation) can be expressed as:

$$\begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} = x_i^T \bar{Q} x_i + v_i^T R v_i$$

where

$$\bar{Q} = Q - \mathbb{S}R^{-1}\mathbb{S}^T$$

In similar way we find for the dynamics

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i \\ &= (A - BR^{-1}\mathbb{S})x_i + Bv_i \\ &= \bar{A}x_i + Bv_i \end{aligned}$$

where

$$\bar{A} = A - BR^{-1}\mathbb{S}$$

For the future cost to go (the second term in the Bellman equation) we have:

$$V_{i+1}(x_{i+1}) = x_{i+1}^T S_{i+1} x_{i+1} = (\bar{A}x_i + Bv_i)^T S_{i+1}(\bar{A}x_i + Bv_i)$$

We have now transformed the problem to the standard form and the inner minimization in the Bellman equation

$$V_i(x_i) = \min_{u_i}\left[x_i^T Q x_i + u_i^T R u_i + V_{i+1}(x_{i+1})\right]$$

is then simply:

$$I = \begin{bmatrix} x_i^T & v_i^T \end{bmatrix} \begin{bmatrix} \bar{Q} + \bar{A}^T S_{i+1}\bar{A} & \bar{A}^T S_{i+1} B \\ B^T S_{i+1}\bar{A} & R + B^T S_{i+1} B \end{bmatrix} \begin{bmatrix} x_i \\ v_i \end{bmatrix}$$

with the solution

$$v_i = -\bar{L}_i x_i \qquad \bar{L}_i = \left[R + B^T S_{i+1} B\right]^{-1} B^T S_{i+1}\bar{A}$$

The candidate function is a solution to the Bellman equation if

$$\begin{aligned} S_i &= \bar{Q} + \bar{A}^T S_{i+1}\bar{A} - \bar{A}^T S_{i+1} B\left[R + B^T S_{i+1} B\right]^{-1} B^T S_{i+1}\bar{A} \\ &= \bar{A}^T S_{i+1}(\bar{A} - B\bar{L}_i) + \bar{Q} \end{aligned} \qquad (14)$$

This means that

$$u_i = -\left[\bar{L}_i + R^{-1}\mathbb{S}\right] x_i \qquad \bar{L}_i = \left[R + B^T S_{i+1} B\right]^{-1} B^T S_{i+1}\bar{A}$$

D.2.2 Direct method

If R is not invertible then we are forced to use a more direct approach which results in the following inner minimization (minimization of the inner part in the Bellman equation):

$$I = \begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} \bar{Q} + \bar{A}^T S_{i+1} \bar{A} & \mathbb{S} + \bar{A}^T S_{i+1} B \\ \mathbb{S}^T + B^T S_{i+1} \bar{A} & R + B^T S_{i+1} B \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix}$$

with the solution

$$u_i = -L_i x_i \qquad L_i = \left[R + B^T S_{i+1} B\right]^{-1} \left[\mathbb{S}^T + B^T S_{i+1} A\right]$$

and a Riccati equation

$$S_i = Q + A^T S_{i+1} A - \left[\mathbb{S} + A^T S_{i+1} B\right] \left[R + B^T S_{i+1} B\right]^{-1} \left[\mathbb{S}^T + B^T S_{i+1} A\right] \tag{15}$$

Notice, that (14) is the standard Riccati equation, whereas (15) contains (directly) the cross term $\mathbb{S}$. The transformation method do require that R is invertible.

D.3 Optimal tracking

In this section we will extend the standard discrete time LQ control problem to include a reference signal. Later on in this presentation, we wil restrict the approach and assume that the reference signal is constant (ie. is a set point). Consider the problem of controlling a dynamic system in discrete time given by (11) such that the cost function

$$J = \|y_N - r_N\|_P^2 + \sum_{i=0}^{N-1} \|Cx_i - r_i\|_Q^2 + \|u_i\|_R^2$$

is minimized. This equivalent to the LQ cost function

$$J = (Cx_N - r_N)^T P (Cx_N - r_N) + \sum_{i=0}^{N-1} (Cx_i - r_i)^T Q (Cx_i - r_i) + u_i^T R u_i \tag{16}$$

is to be minimized.

The Bellman equation will in case be

$$V_i(x_i) = \min_{u_i} \left[(Cx_i - r_i)^T Q (Cx_i - r_i) + u_i^T R u_i + V_{i+1}(x_{i+1})\right] \tag{17}$$

with the end point constraints

$$V_N(x_N) = (Cx_N - r_N)^T P (Cx_N - r_N)$$

If we test the candidate function

$$V_i(x_i) = x_i^T S_i x_i - 2v_i^T x_i + \sigma_i$$

where

$$S_N = C^T P C \qquad v_N = C^T P r_N \qquad \sigma_N = r_N^T P r_N$$

then the inner part of the minimization in (17) will be

$$\begin{aligned} I &= (Cx_i - r_i)^T Q (Cx_i - r_i) + u_i^T R u_i \\ &\quad + (Ax_i + Bu_i)^T S_{i+1} (Ax_i + Bu_i) - 2v_{i+1}^T (Ax_i + Bu_i) + \sigma_{i+1} \\ &= x_i^T (C^T Q C + A^T S_{i+1} A) x_i + u_i^T (R + B^T S_{i+1} B) u_i + r_i^T Q r_i \\ &\quad + 2x_i^T A^T S_{i+1} B u_i - 2(r_i^T Q C + v_{i+1}^T A) x_i - 2v_{i+1}^T B u_i + \sigma_{i+1} \end{aligned}$$

The minimum for the this function is according to Appendix B given by

$$u_i = -\left[R + B^T S_{i+1} B\right]^{-1} \left[B^T S_{i+1} A\ x_i - B^T v_{i+1}\right]$$

and the candidate function is in fact a solution to the Bellman equation in (17) if

$$\begin{aligned}
S_i &= C^T Q C + A^T S_{i+1} A - A^T S_{i+1} B \left[R + B^T S_{i+1} B\right]^{-1} B^T S_{i+1} A \\
v_i &= A^T v_{i+1} + C^T Q r_i - A^T S_{i+1} B \left[R + B^T S_{i+1} B\right]^{-1} B^T v_{i+1} \\
\sigma_i &= \sigma_{i+1} + r_i^T Q r_i - v_{i+1}^T B \left[R + B^T S_{i+1} B\right]^{-1} B^T v_{i+1}
\end{aligned}$$

If we introduce the gains

$$\boxed{L_i = \left[R + B^T S_{i+1} B\right]^{-1} B^T S_{i+1} A \qquad K_i = \left[R + B^T S_{i+1} B\right]^{-1} B^T}$$

then the control law can be written as

$$\boxed{u_i = -L_i x_i + K_i v_{i+1}}$$

and the Ricatti equations becomes

$$\boxed{\begin{aligned}
S_i &= C^T Q C + A^T S_{i+1}(A - BL_i) && S_N = P \\
v_i &= (A - BL_i)^T v_{i+1} + C^T Q r_i && v_N = C^T P r_N \\
\sigma_i &= \sigma_{i+1} + r_i^T Q r_i - v_{i+1}^T B K_i v_{i+1} && \sigma_N = r_N^T P r_N
\end{aligned}}$$

In steady state ($N \to \infty$, $r_i = r$) the solution is given by

$$u_i = -L x_i + K v$$

where

$$L = \left[R + B^T S B\right]^{-1} B^T S A \qquad K = \left[R + B^T S B\right]^{-1} B^T$$

and S is the solution to the algebraic Ricatti equation;

$$S = C^T Q C + A^T S A - A^T S B \left[R + B^T S B\right]^{-1} B^T S A$$

and v found from

$$v = (I - \Phi^T)^{-1} C^T Q r \quad \text{where} \quad \Phi = A - BL$$

Unless the system has (or has been imposed) a pure integration, this strategy will result in a steady state error (for $R > 0$).

D.4 Reference control with cross term

In this section we will extend the result from the previous section a bit further. Consider the problem of controlling a dynamic system in discrete time given as in (11) such that the cost function

$$J = \|\bar{C} x_N - r_N\|_P^2 + \sum_{i=0}^{N} \|z_i - r_i\|_Q^2$$

where

$$z_i = Cx_i + Du_i$$

is minimized. This equivalent to the (standard LQ) cost function

$$\begin{aligned} J &= (\bar{C}x_N - r_N)^T P(\bar{C}x_N - r_N) \\ &\quad + \sum_{i=0}^{N} (Cx_i + Du_i - r_i)^T Q(Cx_i + Du_i - r_i) \end{aligned} \tag{18}$$

is to be minimized. The Bellman equation will in this case be

$$V_i(x_i) = \min_{u_i} \left[(Cx_i + Du_i - r_i)^T Q(Cx_i + Du_i - r_i) + V_{i+1}(x_{i+1}) \right] \tag{19}$$

with the end point constraints

$$V_N(x_N) = (\bar{C}x_N - r_N)^T P(\bar{C}x_N - r_N)$$

If we test the candidate function

$$V_i(x_i) = x_i^T S_i x_i - 2v_i^T x_i + \sigma_i$$

where

$$S_N = \bar{C}^T P \bar{C} \qquad v_N = \bar{C}^T P r_N \qquad \sigma_N = r_N^T P r_N$$

then the inner part of the minimization in (19) will be

$$\begin{aligned} I &= (Cx_i + Du_i - r_i)^T Q(Cx_i + Du_i - r_i) \\ &\quad + (Ax_i + Bu_i)^T S_{i+1}(Ax_i + Bu_i) - 2v_{i+1}^T(Ax_i + Bu_i) + \sigma_i \\ &= x^T(C^TQC + A^TSA)x + r^TQr + u^T(D^TQD + B^TSB)u \\ &\quad + 2x_i^T(C^TQD + A^TSB)u_i - 2(r_i^TQD + v_{i+1}^TB)u_i - 2v_{i+1}^TAx_i + \sigma_{i+1} \end{aligned}$$

The minimum for the this function is according to Appendix B given by

$$u_i = -\left[D^TQD + B^TSB\right]^{-1}\left((D^TQC + B^TSA)\ x_i - B^Tv_i - D^TQr_i\right)$$

If we apply the results in Appendix B then we can see that the candidate function is in fact a solution to the Bellman equation in (25) if

$$\begin{aligned} S_i &= A^TSA + C^TQC - \left(C^TSD + A^TSB\right)\left[D^TQD + B^TSB\right]^{-1}\left(D^TQC + B^TSA\right) \\ v_i &= A^Tv_{i+1} - \left(C^TSD + A^TSB\right)\left[D^TQD + B^TSB\right]^{-1}\left(B^Tv_{i+1} + D^TQr_i\right) \\ \sigma_i &= \sigma_{i+1} + r^TQr - \left(r_i^TQD + v_{i+1}^TB\right)\left[D^TQD + B^TSB\right]^{-1}\left(D^TQr_i + B^Tv_{i+1}\right) \end{aligned}$$

The initial (or rather terminal) conditions are:

$$S_N = C^TPC \qquad v_N = C^TPr_T \qquad \sigma_N = r_T^NPr_N$$

If we introduce the gains

$$\boxed{\begin{array}{c} L_i = \left[D^TQD + B^TSB\right]^{-1}\left(D^TQC + B^TSA\right) \\ K_i = \left[D^TQD + B^TSB\right]^{-1}B^T \qquad M_i = \left[D^TQD + B^TSB\right]^{-1}D^TQ \end{array}}$$

then the control law can be written as

$$\boxed{u_i = -L_ix_i + K_iv_i + M_ir_i}$$

and the Ricatti equations becomes

$$\boxed{\begin{aligned} S_i &= A^T S(A - BL) + C^T QC - C^T SDL \\ v_i &= (A - BL_t)^T v + L^T DQr \\ \sigma_i &= \sigma_{i+1} + r^T Qr - \left(r_i^T QD + v_{i+1}^T B\right)\left[D^T QD + B^T SB\right]^{-1}\left(D^T Qr_i + B^T v_{i+1}\right) \end{aligned}}$$

In steady state ($T \to \infty$, $r_i = r$) the solution is given by

$$u_t = -Lx_t + Kv + Mr$$

where

$$L = \left[D^T QD + B^T SB\right]^{-1}\left(D^T QC + B^T SA\right)$$

$$K = \left[D^T QD + B^T SB\right]^{-1} B^T \qquad M = \left[D^T QD + B^T SB\right]^{-1} D^T Q$$

and S is the solution to the algebraic Ricatti equation;

$$S = A^T SA + C^T QC - \left(C^T SD + A^T SB\right)\left[D^T QD + B^T SB\right]^{-1}\left(D^T QC + B^T SA\right)$$

and v found from

$$v = [I - \Phi^T]^{-1} L^T DQr \quad \text{where} \quad \Phi = A - BL$$

E The Continuous Time LQ Control

In this section we will review the results given in the previous sections but in continuous time. We will start with the standard LQ problem and then in order to connect with the H_2 formulation review the LQ problem with a cross term in the cost function. The problem is to control the LTI system given in continuous time

$$\frac{d}{dt} x_t = Ax_t + Bu_t \qquad x_0 = \underline{x}_0 \tag{20}$$

such that some objectives are meet.

E.1 The Standard CLQ Control problem

Consider the problem of controlling a continuous time LTI system in (20) such that the performance index

$$J = x_T^T P x_T + \int_0^T x_t^T Q x_t + x_t^T Q x_t \; dt$$

is minimized. The Bellman equation is for this situation

$$-\frac{d}{dt} t V_t(x_t) = \min_{u_t} \left[x_t^T Q x_t + u_t^T R u_t + \frac{d}{dt} x V_t(x_t) \left(Ax_t + Bu_t\right) \right]$$

with

$$V_T = x_T^T P x_T$$

as boundary condition. For the candidate function

$$V_t(x_t) = x_t^T S_t x_t$$

this (Bellman) equation becomes

$$-x_t^T \dot{S}_t x_t = \min_{u_t} \left[x_t^T Q x_t + u_t^T R u_t + 2\ x_t^T S_t A x_t + 2\ x_t^T S_t B u_t \right]$$

This is fulfilled for

$$\boxed{u_t = -R^{-1} B^T S_t x_t}$$

The candidate function is indead a Bellman function if S_t is the solution to the Riccati equation

$$-\dot{S}_t = S_t A + A^T S_t + Q - S_t B R^{-1} B^T S_t \qquad S_T = P$$

In terms of the gain

$$L_t = R^{-1} B^T S_t$$

the Riccati equation can also be expressed as

$$\begin{aligned} -\dot{S}_t &= S_t A + A^T S_t + Q - L_t^T R L_t \\ -\dot{S}_t &= S_t (A - B L_t) + A^T S_t + Q = (A - B L_t)^T S_t + S_t A + Q \\ -\dot{S}_t &= S_t (A - B L_t) + (A - B L_t)^T S_t + Q + L_t^T R L_t \end{aligned}$$

It can be shown that

$$J = x_0^T S_0 x_0$$

E.2 CLQ and cross terms

Let us now focus on the problem where performance index has a cross term, i.e. where

$$J = x_T^T P x_T + \int_0^T x_t^T Q x_t + x_t^T Q x_t + 2 x_t^T \mathbb{S} u_t\ dt$$

As in the discrete time case this will typically be the case if the problem arise from a minimization of

$$J = \int_0^T |y_t|_W^2\ dt$$

which is a weighted (W) integral square of the output

$$y_t = C x_t + D u_t$$

i.e. the H_2 problem. In that case

$$\begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} W \begin{bmatrix} C & D \end{bmatrix}$$

The Bellman equation is now for this situation

$$-\frac{d}{dt} t V_t(x_t) = \min_{u_t} \left[\begin{bmatrix} x_t^T & u_t^T \end{bmatrix} \begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} + \frac{d}{dt} x V_t(x_t) \left(A x_t + B u_t \right) \right] \tag{21}$$

with

$$V_T = x_T^T P x_T$$

as boundary condition. Again we can go directly for a solution, but if R is invertible, we can tranform the problem to the standard form.

E.2.1 Using transformation technique

If we use the same method as in the discrete time and introduce a new decision variable, v_t through

$$u_t = v_t - R^{-1}\mathbb{S}^T x_t$$

then instantanuous loss term is rewritten to

$$\begin{bmatrix} x_t^T & u_t^T \end{bmatrix} \begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} = x_t^T \bar{Q} x_t + v_t^T R v_t$$

where

$$\bar{Q} = Q - \mathbb{S}R^{-1}\mathbb{S}^T$$

Furthermore the dynamics is transformed to

$$Ax_t + Bu_t = (A - BR^{-1}\mathbb{S}^T)x_t + Bv_t = \bar{A}x_t + Bv_t$$

where

$$\bar{A} = (A - BR^{-1}\mathbb{S}^T)$$

The Bellman equation is now in the newly introduced variable

$$-\frac{d}{dt}tV_t(x_t) = \min_{v_t} \left[x_t^T \bar{Q} x_t + v_t^T R v_t + \frac{d}{dt}x V_t(x_t) \left(\bar{A}x_t + Bv_t \right) \right]$$

with

$$V_T = x_T^T P x_T$$

as boundary condition. For the candidate function

$$V_t(x_t) = x_t^T S_t x_t$$

this Bellman equations becomes

$$-x_t^T \dot{S}_t x_t = \min_{v_t} \left[x_t^T \bar{Q} x_t + v_t^T R v_t + 2\ x_t^T S_t \bar{A} x_t + 2\ x_t^T S_t B v_t \right]$$

The solution to this problem is

$$v_t = -\bar{L}_t x_t \qquad \bar{L}_t = R^{-1}B^T S_t$$

where

$$\boxed{-\dot{S}_t = S_t\bar{A} + \bar{A}^T S_t + \bar{Q} - S_t B R^{-1} B^T S_t \qquad S_T = P} \tag{22}$$

The last equation ensures that the candidate function indeed is a solution. The total solution is consequently given as

$$\boxed{u_t = -(\bar{L}_t + R^{-1}\mathbb{S}^T)x_t \qquad \bar{L}_t = R^{-1}B^T S_t}$$

or simpy as

$$u_t = -R^{-1}(B^T S_t + \mathbb{S}^T)x_t$$

Notice, that (22) is the same Riccati equation that arise from the standard problem except for the transformation of A and Q. Furthermore $\bar{L}$ is the same as arise from the standard problem.

E.2.2 Direct method

If R is not invertible then (21) must be solved directly. For the candidate function

$$V_t(x_t) = x_t^T S_t x_t$$

the Bellman equation, (21), becomes

$$-x_t^T \dot{S}_t x_t = \min_{u_t} \left[x_t^T Q x_t + u_t^T R u_t + 2 x_t^T \mathbb{S} u_t + 2\ x_t^T S_t A x_t + 2\ x_t^T S_t B u_t \right]$$

which is minimized for

$$\boxed{u_t = -R^{-1}(B^T S_t + \mathbb{S}^T) x_t}$$

where

$$\boxed{-\dot{S}_t = S_t A + A^T S_t + Q - (S_t B + \mathbb{S}) R^{-1} (B^T S_t + \mathbb{S}^T) \qquad S_T = P \qquad (23)}$$

It is quite easy to check that (for R being invertible) the solutions to (23) and (22) are identical.

E.3 Reference control

In this section we will extend the standard discrete time LQ control problem to include a reference signal. Consider the problem of controlling a dynamic system in continuous time given by (20) such that the cost function

$$J = \|y_T - r_T\|_P^2 + \int_0^T \|Cx_t - r_t\|_Q^2 + \|u_t\|_R^2 \ \ dt$$

is minimized. This equivalent to the (standard LQ) cost function

$$J = (Cx_T - r_T)^T P (Cx_T - r_T) + \int_0^T (Cx_t - r_t)^T Q (Cx_t - r_t) + u_t^T R u_t \ dt \qquad (24)$$

is to be minimized. The Bellman equation will in this case be

$$-\frac{d}{dt} t V_t(x_t) = \min_{u_t} \left[(Cx_t - r_t)^T Q (Cx_t - r_t) + u_t^T R u_t + \frac{d}{dt} x V_t(x_t)(Ax_t + Bu_t) \right] \qquad (25)$$

with the end point constraints

$$V_T(x_T) = (Cx_T - r_T)^T P (Cx_T - r_T)$$

If we test the candidate function

$$V_t(x_t) = x_t^T S_t x_t - 2 v_t^T x_t + \sigma_t$$

where

$$S_T = C^T P C \qquad v_T = C^T P r_T \qquad \sigma_T = r_T^T P r_T$$

then the inner part of the minimization in (25) will be

$$\begin{aligned} I &= (Cx_t - r_t)^T Q (Cx_t - r_t) + u_t^T R u_t + 2\left[x^T S - v^T \right] (Ax + Bu) \\ &= x^T (C^T Q C + AS + SA^T) x + r^T Q r + u^T R u \\ &- \ \ 2\left[r^T Q C + v^T A \right] x_t + 2(x^T S - v^T) B u_t \end{aligned}$$

The minimum for the this function is according to Appendix A given by

$$u_t = -R^{-1}B^T\left[S_t\ x_t - v_t\right]$$

The candidate function is in fact a solution to the Bellman equation in (25) if

$$\begin{aligned} -\dot{S}_t &= S_tA + A^TS_t + C^TQC - S_tBR^{-1}B^TS_t & S_T &= C^TPC \\ -\dot{v} &= (A - BR^{-1}B^TS)^Tv + C^TQr & v_T &= C^TPr_T \\ -\dot{\sigma} &= r^TQr - v^TBR^{-1}B^Tv & \sigma_T &= r_T^TPr_T \end{aligned}$$

If we introduce the gains

$$\boxed{L_t = R^{-1}B^TS_t \qquad\qquad K_t = R^{-1}B^T}$$

then the control law can be written as

$$\boxed{u_t = -L_tx_t + K_tv_t}$$

and the Ricatti equations becomes

$$\boxed{\begin{aligned} -\dot{S}_t &= S_tA + A^TS_t + C^TQC - L_t^TRL_t & S_T &= C^TPC \\ -\dot{v} &= (A - BL_t)^Tv + C^TQr & v_T &= C^TPr_T \\ -\dot{\sigma} &= r^TQr - v^TK_t^TRK_tv_t & \sigma_T &= r_T^TPr_T \end{aligned}}$$

In steady state ($T \to \infty$, $r_t = r$) the solution is given by

$$u_t = -Lx_t + Kv$$

where

$$L = R^{-1}B^TS_t \qquad\qquad K = R^{-1}B^T$$

and S is the solution to the algebraic Ricatti equation;

$$0 = SA + A^TS + C^TQC - SBR^{-1}B^TS$$

and v found from

$$v = \Phi^{-T}C^TQr \quad \text{where} \quad \Phi = A - BL$$

E.4 Reference control with cross term

In this section we will extend the results related to reference control and include a possible cross term. Consider the problem of controlling a dynamic system in continuous time given by (20) such that the cost function

$$J = \|\bar{C}x_T - r_T\|_P^2 + \int_0^T \|z_t - r_t\|_Q^2 \ dt$$

where

$$z_t = Cx_t + Du_t$$

is minimized. This equivalent to the (standard LQ) cost function

$$J = (\bar{C}x_T - r_T)^TP(\bar{C}x_T - r_T) + \int_0^T (Cx_t + Du_t - r_t)^TQ(Cx_t + Du_t - r_t)\ dt \quad (26)$$

is to be minimized.

The Bellman equation will in case be

$$-\frac{d}{dt}tV_t(x_t) = \min_{u_t}\left[(Cx_t + Du_t - r_t)^T Q(Cx_t + Du_t - r_t) + \frac{d}{dt}xV_t(x_t)(Ax_t + Bu_t)\right] \tag{27}$$

with the end point constraints

$$V_T(x_T) = (\bar{C}x_T - r_T)^T P(\bar{C}x_T - r_T)$$

If we test the candidate function

$$V_t(x_t) = x_t^T S_t x_t - 2v_t^T x_t + \sigma_t$$

where

$$S_T = \bar{C}^T P\bar{C} \qquad v_T = \bar{C}^T Pr_T \qquad \sigma_T = r_T^T Pr_T$$

then the inner part of the minimization in (27) will be

$$\begin{aligned}
I &= (Cx_t + Du_t - r_t)^T Q(Cx_t + Du_t - r_t) + 2\left[x^T S - v^T\right](Ax + Bu) \\
&= x^T(C^T QC + AS + SA^T)x + r^T Qr + u^T D^T QDu \\
&\quad + \; 2x^T C^T QDu_t - 2r_t^T QDu_t \\
&\quad - \; 2\left[r^T QC + v^T A\right]x_t + 2(x^T S - v^T)Bu_t \\
&= x^T(C^T QC + AS + SA^T)x + r^T Qr - 2\left[r^T QC + v^T A\right]x_t \\
&\quad +u^T D^T QDu + 2x^T C^T QDu_t - 2r_t^T QDu_t + 2(x^T S - v^T)Bu_t
\end{aligned}$$

The minimum for the this function is according to Appendix A given by

$$u_t = -\left[D^T QD\right]^{-1}\left((D^T QC + B^T S_t)\; x_t - B^T v_t - D^T Qr_t\right)$$

The candidate function is in fact a solution to the Bellman equation in (25) if

$$\begin{aligned}
-\dot{S}_t &= S_t A + A^T S_t + C^T QC - (C^T QD + SB)(D^T QD)^{-1}(D^T QC + B^T S_t) \\
-\dot{v} &= (A - B(D^T QD)^{-1}(D^T QC + B^T S_t))^T v + C^T Qr - (C^T QD + SB)(D^T QD)^{-1}D^T Qr \\
-\dot{\sigma} &= r^T Qr - (v^T B + r^T QD)(D^T QD)^{-1}(B^T v_t + D^T Qr_t)
\end{aligned}$$

The terminal conditions are:

$$S_T = C^T PC \qquad v_T = C^T Pr_T \qquad \sigma_T = r_T^T Pr_T$$

If we introduce the gains

$$\boxed{\begin{gathered} L_t = \left[D^T QD\right]^{-1}(D^T QC + B^T S_t) \\ K_t = \left[D^T QD\right]^{-1}B^T \qquad\qquad M_t = \left[D^T QD\right]^{-1}D^T Q \end{gathered}}$$

then the control law can be written as

$$\boxed{u_t = -L_t x_t + K_t v_t + M_t r_t}$$

and the Ricatti equations becomes

$$\boxed{\begin{array}{rcll} -\dot{S}_t & = & S_tA + A^TS_t + C^TQC - L_t^T(D^TQD)L_t & S_T = C^TPC \\ -\dot{v} & = & (A - BL_t)^Tv + (C - DL)^TQr & v_T = C^TPr_T \\ -\dot{\sigma} & = & r^TQr - (v_t^TK_t^T + r_t^TM^T)(D^TQD)(K_tv_t + M_tr_t) & \sigma_T = r_T^TPr_T \end{array}}$$

In steady state ($T \to \infty$, $r_t = r$) the solution is given by

$$u_t = -Lx_t + Kv + Mr$$

where

$$L_t = \left[D^TQD\right]^{-1}(D^TQC + B^TS_t)$$

$$K_t = \left[D^TQD\right]^{-1}B^T \qquad M_t = \left[D^TQD\right]^{-1}D^TQ$$

and S is the solution to the algebraic Ricatti equation;

$$0 = S_tA + A^TS_t + C^TQC - (C^TQD + SB)(D^TQD)^{-1}(D^TQC + B^TS_t)$$

and v found from

$$v = \Phi^{-T}(C - DL)^TQr \quad \text{where} \quad \Phi = A - BL$$